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Scaling and Expansion of Moment Equations in Kinetic Theory

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The set of generalized 13 moment equations for molecules interacting with power law potentials [Struchtrup, *Multiscale Model. Simul*. **3**:211 (2004)] forms the base for an investigation of expansion methods in the Knudsen number and other scaling parameters. The scaling parameters appear in the equations by introducing dimensionless quantities for all variables and their gradients. Only some of the scaling coefficients can be chosen independently, while others depend on these chosen scales–their size can be deduced from a Chapman–Enskog expansion, or from the principle that a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms.

It is shown that for the least restrictive scaling the new order of magnitude expansion method [Struchtrup, *Phys. Fluids* **16**(11):3921 (2004)] reproduces the original equations after only two expansion steps, while the classical Chapman–Enskog expansion would require an infinite number of steps. Both methods yield the Euler and Navier–Stokes–Fourier equations to zeroth and first order. More restrictive scaling choices, which assume slower time scales, small velocities, or small gradients of temperature, are considered as well.

KEY WORDS: Kinetic theory, moment methods, Chapman–Enskog, expansion, Burnett equations

1. INTRODUCTION

This paper investigates expansions of the Boltzmann equation(1) and its moment equations in terms of suitable smallness parameters. The goal of these methods is to derive macroscopic transport equations.(2)

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The choice of the smallness parameters is crucial, and some emphasis lies on the discussion of the scales that can—or must—be chosen for the various terms that arise in the equations.

The standard method to derive macroscopic equations is the Chapman–Enskog (CE) method, developed some 80 years ago independently by Enskog[3,4] and Chapman.[5,6] The CE method employs an expansion in the Knudsen number to derive the Euler equations at zeroth order, the Navier–Stokes–Fourier (NSF) equations at first order, and the Burnett[7] and super-Burnett[8] equations at second and third order, respectively. Euler and NSF equations are nowadays standard tools, and their validity is not to be questioned. The Burnett and super-Burnett equations, however, suffer from instabilities[9,10] and never gained wide acceptance as a tool for the description of rarefied gas flows. Recently, several authors presented regularization methods for the Burnett equations which yield stable equations at second order.[11–13]

While Enskog based his expansion on the Boltzmann equation,[3,4] Chapman considered expansions of the corresponding moment equations,[5,6] but later, in his famous book with Cowling, he adopted Enskog's approach.[14] Most presentations in textbooks follow this lead, e.g. see Ref. 15–18.

The idea to consider the moment equations was revived in the work of Reinecke and Kremer[19–21] who presented a method that combines ideas from the Chapman–Enskog expansion with Grad's moment method.[22,23] The results are identical to those from the Chapman–Enskog method, and thus this method does not remove the problems with the higher order equations, i.e. the instabilities.

Only recently the present author introduced the so-called *Order of Magnitude Approach* to derive approximations to the Boltzmann equation from its infinite set of corresponding moment equations.[2,24,25] This method first determines the order of magnitude of all moments by means of a Chapman–Enskog expansion, forms linear combinations of moments in order to have the minimum number of moments at a given order, and then uses the information on the order of the moments to properly rescale the moment equations. The rescaled moment equations are finally systematically reduced by cancelling terms of higher order. The idea of rescaling moment equations was first discussed by Müller *et al.* in Ref. 26 where they presented a similar, but different method for the reduction of the equations; see also Refs. 2,24,25 for further discussion.

The order of magnitude approach gives the Euler and NSF equations at zeroth and first order, and thus agrees with the Chapman–Enskog method in the lower orders. At higher orders, however, the resulting equations are quite different: The second order equations turn out to be the well-known 13 moment equations of Grad[22,23] for Maxwell molecules, and an extension of these for molecules that interact with power potentials. At third order, the method was only performed for Maxwell molecules, where it yields a regularization of the Grad equations, the so-called Regularized 13 (R13) moment equations. These were first found by

combining the methods of Grad and Chapman–Enskog in Refs. 27,28 see also Grad's work[23] for a first step into the same direction. Most notably, the resulting higher order equations are stable, and they contain the Burnett and super-Burnett equations, since these can be obtained by means of a Chapman–Enskog expansion of the R13 equations.[27,28]

Due to the complexity of the infinite system of moment equations of the Boltzmann equation, the original papers[24,25] presenting the order of magnitude approach are not easy accessible. One aim of the present paper is to present the method in a more accessible frame, by applying it to a finite set of moment equations, namely the set of 13 moment equations that result as the proper second order equations for power potentials. For these, the method becomes more transparent, and seems surprisingly simple. The order of magnitude approach reproduces the original equations after only few steps, while in a similar problem the Chapman–Enskog method requires an infinite number of expansion coefficients to reproduce the original equations.[29,30]

The simpler problem allows discussion of the importance of the proper scaling of the equations, which is obtained by introducing dimensionless quantities. Working on the level of the moment equations allows introduction of scales for space and time, as well as separate scales for all variables and their gradients. In contrast to this, working on the level of the Boltzmann equation allows introduction of comparatively few scaling parameters. Thus, the scaling of the moment equations gives a higher flexibility within the scaling process.

At first, the scaling introduces a large number of scaling coefficients into the equations. Our discussion will show that only some of these parameters can be chosen independently, while others depend on these chosen scales–their size can be deduced from the transport equations. The size of the dependent scaling parameters follows from the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms.* Thus, the size of the dependent scaling parameters is chosen as the maximum possible size; the result stands in agreement with the order of magnitude that can be deduced from a Chapman–Enskog expansion.

Only the proper scaling allows application of the order of magnitude method, and its application is demonstrated for two particular scaling choices, the classical Chapman–Enskog scaling and a scaling appropriate for diffusion problems in slow non-isothermal flows. The latter was considered by Bobylev by means of Chapman–Enskog expansions in Ref. 31 and the application of the order of magnitude method yields essentially the same results. The emphasis of the present paper rather lies on the method of derivation, and not on the equations themselves.

A detailed discussion of slow flows can be found in the recent book by Sone[32] and the references therein. The development of macroscopic transport equations from the Boltzmann equation is a special case of coarse-graining, and

we refer the reader to Öttinger's recent book[33] for a detailed discussion of other coarse-graining procedures.

The paper starts with the set of 13 moment equations which are presented in Sec. 2. Then we introduce the basic dimensionless quantities for the thirteen variables and their gradients, and the corresponding scaling parameters in Sec. 3. The classical Chapman–Enskog (CE) expansion method is discussed in Sec. 4. It relies on a simple scaling where scaling of the gradients is not necessary. From the CE expansion it becomes clear that the order of magnitude of higher moments (stress, heat flux) follows from the choice of the basic scales for time and length, and for temperature, density and velocity. Section 5 introduces an alternative (but equivalent) approach for determining the order of magnitude of higher moments. The result allows to properly rescale the equations, which then are subjected to the order of magnitude method. The results of both methods will be compared.

The CE scaling is least restrictive, since it assumes smallness only of the Knudsen number but allows large values for all other scaling parameters. In particular, it includes fast processes, large velocities, and large gradients. Additional knowledge on scales—including those of gradients—allows further reduction of the equations, and in Sec. 6 an example is given by a diffusion scaling for slow non-isothermal flows.[31,34] The order of magnitude approach is used to retrieve the proper set of equations under the prescribed scaling at various orders. The paper ends with our conclusions.

2. 13 MOMENT EQUATIONS FOR POWER POTENTIALS

We shall not consider the full infinite system of moment equations of kinetic theory, but the set of 13 moment equations for power potentials which was derived in Refs. 11,12 based on the order of magnitude method introduced in Ref. 24 Indeed, our goal is to clarify the arguments of this method, and this is easier for a finite system of equations.

The equations presented below agree to second order with those introduced and discussed in Refs. 11,12 which resulted from arguments on the regularization of the Burnett equations, see Ref. 2 for a more detailed discussion.

In our notation, the conservation laws for mass, momentum and energy read

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial v_k}{\partial x_k} &= 0, \\ \rho \frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k} &= 0, \\ \frac{3}{2}\rho \frac{\partial \theta}{\partial t} + \frac{3}{2}\rho v_k \frac{\partial \theta}{\partial x_k} + p \frac{\partial v_k}{\partial x_k} + \sigma_{ij} \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \frac{\partial q_k}{\partial x_k} &= 0, \end{aligned} \tag{1}$$

where ρ denotes the mass density, v_k is the velocity, and $\theta = RT$ is the temperature in energy units (T is the absolute temperature and R is the gas constant). The pressure p obeys the ideal gas law, $p = \rho\theta$. Indices in angular brackets indicate the symmetric and traceless part of a tensor.

The conservation laws do not form a closed system for ρ, v_i, θ, since they also contain the stress tensor σ_{ij} (i.e. the deviatoric part of the pressure tensor with $\sigma_{ij} = \sigma_{\langle ij\rangle}$, $\sigma_{kk} = 0$) and the heat flux vector q_k. The order of magnitude method applied to the infinite system of moment equations finds the equations for these as (with $S_{jk} = \frac{\partial v_{\langle j}}{\partial v_{k\rangle}}$)[2]

$$\begin{aligned}\frac{\partial \sigma_{ij}}{\partial t} &+ v_k \frac{\partial \sigma_{ij}}{\partial x_k} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_3}{\varpi_2}\left(\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - \omega q_{\langle i}\frac{\partial \ln\theta}{\partial x_{j\rangle}}\right)\\ &+ \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_4}{\varpi_2} q_{\langle i}\frac{\partial \ln p}{\partial x_{j\rangle}} + \frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_5}{\varpi_2} q_{\langle i}\frac{\partial \ln\theta}{\partial x_{j\rangle}} + \left(\frac{\varpi_6}{\varpi_2} - 4\right)\sigma_{k\langle i}S_{j\rangle k}\\ &+ \frac{4}{\varpi_2} p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{2}{\varpi_2}\frac{p}{\mu}\sigma_{ij}, \end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial q_i}{\partial t} &+ v_k\frac{\partial q_i}{\partial x_k} + q_k\frac{\partial v_i}{\partial x_k} + \frac{5}{3}q_i\frac{\partial v_k}{\partial x_k} + \frac{5}{2}\frac{1}{\mathrm{Pr}}\left(\frac{3}{2}\frac{\theta_5}{\theta_2} - 1\right)\sigma_{ik}\frac{\partial\theta}{\partial x_k}\\ &+ \frac{5}{4}\frac{1}{\mathrm{Pr}}\frac{\theta_3}{\theta_2}\theta\sigma_{ik}\frac{\partial \ln p}{\partial x_k} + \frac{5}{4}\frac{1}{\mathrm{Pr}}\frac{\theta_4}{\theta_2}\theta\left(\frac{\partial\sigma_{ik}}{\partial x_k} - \omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k}\right)\\ &+ \frac{1}{\theta_2}\frac{25}{4}\frac{1}{\mathrm{Pr}^2}p\frac{\partial\theta}{\partial x_i} = -\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}q_i. \end{aligned} \tag{3}$$

Here, ϖ_α and θ_α denote the Burnett coefficients, $\mathrm{Pr} \simeq 2/3$ is the Prandtl number, and

$$\mu = \mu_0\left(\frac{\theta}{\theta_0}\right)^{\omega} \tag{4}$$

denotes the viscosity, where μ_0 is the viscosity at reference temperature θ_0, and ω is the viscosity exponent that depends on the degree of the interaction potential. The Burnett coefficients depend on the molecule type, and for power potentials (4) they are given in Table I,[19] which also shows the corresponding viscosity exponent ω.

Altogether, the Eqs. (1)–(3) form a set of 13 coupled partial differential equations for the 13 variables $\{\rho, \theta, v_k, \sigma_{ij}, q_k\}$.

Table I. Burnett coefficients for power potentials ($\gamma = 5$ for Maxwell molecules, $\gamma = \infty$ for for hard sphere molecules). [19]

γ	ω	ϖ_1	ϖ_2	ϖ_3	ϖ_4	ϖ_5	ϖ_6	θ_1	θ_2	θ_3	θ_4	θ_5
5	1	10/3	2	3	0	3	8	75/8	45/8	−3	3	39/4
7	0.833	3.561	2.003	2.793	0.217	1.942	7.781	10.038	5.647	−3.010	2.793	9.113
7.66	0.8	3.600	2.004	2.761	0.254	1.784	7.748	10.160	5.656	−3.014	2.761	9.019
9	0.75	3.679	2.007	2.695	0.328	1.466	7.681	10.402	5.674	−3.023	2.695	8.829
17	0.625	3.863	2.016	2.553	0.500	0.814	7.543	10.995	5.736	−3.053	2.553	8.442
∞	0.5	4.056	2.028	2.418	0.681	0.219	7.424	11.644	5.822	−3.09	2.418	8.286

3. SCALES AND DIMENSIONLESS EQUATIONS

3.1. Basic Scales

For a proper scaling procedure it is best to introduce dimensionless quantities and coefficients. For this, we introduce a reference length x_0 and a reference time t_0, and the reference values for the variables as ρ_0, θ_0, $p_0 = \rho_0\theta_0$, v_0, σ_0 and q_0. Dimensionless quantities are defined as, e.g., $\hat{x} = x/x_0$, $\hat{t} = t/t_0$, $\hat{\rho} = \rho/\rho_0$, etc. We emphasize that each of the basic variables is made dimensionless by its own reference quantity, which defines the magnitude of the respective variable.

Before we proceed, it is worthwhile to discuss the choice of the reference values:

The reference length x_0 depends on the size of the system considered, or the desired spatial resolution. Typical choices are, e.g., the width of a channel, the diameter of an obstacle, or the smallest wavelength to be resolved.

The reference time t_0 depends on the desired time resolution. Typical choices would be the response time of a measurement device, the inverse of the largest frequency that needs to be observed, or the time a signal requires to travel the reference length x_0.

Reference density ρ_0, temperature θ_0, and velocity v_0 are suitable average values that describe the gas. Normally they will be known from the prescribed initial, boundary, and inflow conditions. These three values refer to the conserved variables mass, momentum and energy, which all are non-zero in equilibrium.

Stress and heat flux depend on the response of the gas to the prescribed initial and boundary conditions. They are non-equilibrium quantities, and thus finding proper reference values is a non-trivial task. Indeed, when velocity and temperature are prescribed at the boundary, it is impossible to independently control stresses and heat flux, which must be adjusted such that the desired values of velocity and temperature are realized. In that sense they are dependent on temperature and velocity (or rather their gradients), and the same can be said about their reference values.

The scaling procedure introduces five dimensionless combinations of the reference values into the 13 moment equations which are

$$\mathrm{Kn} = \frac{\mu_0\sqrt{\theta_0}}{p_0 x_0}, \quad \varepsilon_t = \frac{x_0}{t_0\sqrt{\theta_0}}, \quad \varepsilon_v = \frac{v_0}{\sqrt{\theta_0}}, \quad \varepsilon_\sigma = \frac{\sigma_0}{p_0}, \quad \varepsilon_q = \frac{q_0}{p_0\sqrt{\theta_0}}. \tag{5}$$

Here, $\mathrm{Kn} = l/x_0$ is the Knudsen number of the problem that relates the relevant lengthscale x_0 to the mean free path $l = \mu_0\sqrt{\theta_0}/p_0$. The Knudsen number is the usual smallness parameter of the CE method.

The coefficient ε_t relates the scales of time and space. The CE method assumes $\varepsilon_t = 1$, which means that time is resolved on a scale defined by the time a soundwave needs to cover the distance x_0. For diffusive problems it suffices to consider a time scale where the time unit is much larger; then ε_t becomes small.

The coefficient ε_v is a measure for the Mach number $\mathrm{Ma} = v_0/\sqrt{5/3}\,\theta_0 = \sqrt{3/5}\,\varepsilon_v$, and thus gives an indication of how fast the flow moves with respect to a properly chosen frame of reference (e.g. the laboratory frame). The CE method normally assumes $\varepsilon_v = 1$, so that fast flow is included. For slow flows, however, ε_v can be assumed as being small.

Finally, ε_σ and ε_q are measures for the magnitude of stress and heat flux. These normally do not appear in the CE method which means that the CE method assumes $\varepsilon_\sigma = \varepsilon_q = 1$.

One of the points of this paper is that the CE expansion allows to determine the sizes of ε_σ and ε_q as a result of the choice of the basic scales Kn, ε_t, ε_v. The order of magnitude approach then considers the values of all five parameters (5) to determine the proper equations at a given order.

Note that the classical CE expansion of the Boltzmann equation does not allow consideration of the magnitude of the moments of the phase density, so that parameters like ε_σ and ε_q do not appear. Since the present work concerns the moment equations, it is natural to introduce scales for *all* moments.

In an even more refined scaling process, one could introduce different reference values for the components of velocity, stress, and heat flux. Indeed, the values of the components can be quite different, as a result of boundary conditions, symmetry and geometry of a particular problem. Therefore, the proper scaling depends on the particular problem under consideration. Our subsequent discussion will be based solely on the transport equations, and thus we cannot differentiate between the components. The resulting scales are the maximum scales for the components, and one or more of the components can have a smaller order of magnitude than the one indicated. A scaling of the elements of vector and tensor quantities can help to further reduce the equations, see, e.g. Ref. 26 or Chap. 12 of Ref. 2 where it is shown that in Couette flow some components of stress and heat flux are of first order in Kn while others are of second order.

3.2. Scaling Gradients

The variables and their gradients can be scaled independently. Indeed, in many problems one can expect small gradients, and then it is more natural to scale the gradients by means of a relevant difference. For instance the temperature gradient should be scaled as

$$\frac{\partial\theta}{\partial x_i} \to \frac{\Delta\theta_0}{x_0}\frac{\partial\hat{\theta}}{\partial\hat{x}_i} = \frac{\Delta\theta_0}{\theta_0}\frac{\theta_0}{x_0}\frac{\partial\hat{\theta}}{\partial\hat{x}_i} = \varepsilon_{\Delta\theta}\frac{\theta_0}{x_0}\frac{\partial\hat{\theta}}{\partial\hat{x}_i},$$

where $\varepsilon_{\Delta\theta} = \frac{\Delta\theta_0}{\theta_0}$. Similar arguments can be used for the other variables, and this leads to additional scaling coefficients

$$\varepsilon_{\Delta\rho},\quad \varepsilon_{\Delta\theta},\quad \varepsilon_{\Delta p},\quad \varepsilon_{\Delta v},\quad \varepsilon_{\Delta\sigma},\quad \varepsilon_{\Delta q}. \tag{6}$$

Obviously, one needs to find the proper values for these coefficients, which should depend on boundary conditions and/or the response of the system to these (as described through the equations).

In the following we shall set

$$\varepsilon_{\Delta v} = \varepsilon_{\Delta\sigma} = \varepsilon_{\Delta q} = 1. \tag{7}$$

This implies that the relevant differences Δv_0, $\Delta\sigma_0$, Δq_0 are of the same magnitude as the reference values v_0, σ_0, q_0. Indeed, when the velocity is measured against a meaningful frame of reference within the process (say a wall at rest), one will expect velocities between zero and the maximum velocity, and v_0 will be a suitable measure for the mean velocity *and* the reference velocity difference Δv_0. Also, when the gas is somewhere within the system at, or close to, equilibrium (E), where stress and heat flux vanish, then $\Delta\sigma = \sigma_0 - \sigma_E = \sigma_0$, $\Delta q = q_0 - q_E = q_0$.

The remaining three parameters $\varepsilon_{\Delta\rho}, \varepsilon_{\Delta\theta}, \varepsilon_{\Delta p}$ are not independent: From the ideal gas law, $p = \rho\theta$, we find in non-dimensional form

$$\varepsilon_{\Delta p}\frac{\partial \hat{p}}{\partial \hat{x}_i} = \varepsilon_{\Delta\theta}\hat{\rho}\frac{\partial \hat{\theta}}{\partial \hat{x}_i} + \varepsilon_{\Delta\rho}\hat{\theta}\frac{\partial \hat{\rho}}{\partial \hat{x}_i}.$$

Since a single term in this equation cannot be larger by one order of magnitude than the other two, it follows that

$$\varepsilon_{\Delta\rho} = \begin{cases} = \max[\varepsilon_{\Delta p}, \varepsilon_{\Delta\theta}], & \text{if } \varepsilon_{\Delta p} \neq \varepsilon_{\Delta\theta} \\ \leq \varepsilon_{\Delta\theta}, & \text{if } \varepsilon_{\Delta p} = \varepsilon_{\Delta\theta} \end{cases} \tag{8}$$

Thus, only two of the three additional scaling parameters can be chosen independently. More knowledge on these parameters will follow from assigning values to some of the other scaling parameters, as will be seen in the course of our discussion.

It seems natural to assume that the scaling parameters $\varepsilon_{\Delta\rho}$, $\varepsilon_{\Delta\theta}$, $\varepsilon_{\Delta p}$ are not larger than unity.

In principle, it would be possible to rescale the time derivatives as well, by introducing yet another set of parameters, e.g. $\varepsilon^{(t)}_{\Delta\phi} = \frac{\Delta\phi_t}{\phi_0}$ for a variable ϕ, where $\Delta\phi_t$ is a measure for the change of ϕ over time. In the following, we assume that the time derivatives need not to be rescaled, so that

$$\frac{\partial \phi}{\partial t} \to \frac{\phi_0}{t_0}\frac{\partial \hat{\phi}}{\partial t} = \varepsilon_t \frac{\phi_0\sqrt{\theta_0}}{x_0}\frac{\partial \hat{\phi}}{\partial t}.$$

Then, ε_t is the only relevant parameter for the scaling of time derivatives.

3.3. Dimensionless Equations

The dimensionless form of the Eqs. (1)–(3) follows by simple manipulations, which introduce the scaling parameters Eqs. (5)–(7). To simplify the notation, the "hats" on the dimensionless variables are omitted, so that the dimensionless scaled equations read

$$\varepsilon_t \frac{\partial \rho}{\partial t} + \varepsilon_v \varepsilon_{\Delta\rho} v_k \frac{\partial \rho}{\partial x_k} + \varepsilon_v \rho \frac{\partial v_k}{\partial x_k} = 0,$$

$$\varepsilon_t \varepsilon_v \rho \frac{\partial v_i}{\partial t} + \varepsilon_v^2 \rho v_k \frac{\partial v_i}{x_k} + \varepsilon_{\Delta p} \frac{\partial p}{\partial x_i} + \varepsilon_\sigma \frac{\partial \sigma_{ik}}{\partial x_k} = 0, \quad (9)$$

$$\varepsilon_t \frac{3}{2} \rho \frac{\partial \theta}{\partial t} + \varepsilon_v \varepsilon_{\Delta\theta} \frac{3}{2} \rho v_k \frac{\partial \theta}{\partial x_k} + \varepsilon_v \rho \theta \frac{\partial v_k}{\partial x_k} + \varepsilon_v \varepsilon_\sigma \sigma_{ij} \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \varepsilon_q \frac{\partial q_k}{\partial x_k} = 0,$$

$$\varepsilon_t \varepsilon_\sigma \frac{\partial \sigma_{ij}}{\partial t} + \varepsilon_v \varepsilon_\sigma \left[v_k \frac{\partial \sigma_{ij}}{\partial x_k} + \sigma_{ij} \frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i} \frac{\partial v_{j\rangle}}{\partial x_k} + \left(\frac{\varpi_6}{\varpi_2} - 4 \right) \sigma_{k\langle i} S_{j\rangle k} \right]$$
$$+ \varepsilon_q \frac{4}{5} \mathrm{Pr} \left[\frac{\varpi_3}{\varpi_2} \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - \varepsilon_{\Delta\theta} \frac{\varpi_3}{\varpi_2} \omega q_{\langle i} \frac{\partial \ln \theta}{\partial x_{j\rangle}} + \varepsilon_{\Delta p} \frac{\varpi_4}{\varpi_2} q_{\langle i} \frac{\partial \ln p}{\partial x_{j\rangle}} + \varepsilon_{\Delta\theta} \frac{\varpi_5}{\varpi_2} q_{\langle i} \frac{\partial \ln \theta}{\partial x_{j\rangle}} \right]$$
$$+ \varepsilon_v \frac{4}{\varpi_2} p \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} = -\frac{\varepsilon_\sigma}{\mathrm{Kn}} \frac{2}{\varpi_2} \frac{p}{\mu} \sigma_{ij}, \quad (10)$$

$$\varepsilon_t \varepsilon_q \frac{\partial q_i}{\partial t} + \varepsilon_q \varepsilon_v \left[v_k \frac{\partial q_i}{\partial x_k} + q_k \frac{\partial v_i}{\partial x_k} + \frac{5}{3} q_i \frac{\partial v_k}{\partial x_k} \right]$$
$$+ \varepsilon_\sigma \frac{5}{2} \frac{1}{\mathrm{Pr}} \left[\varepsilon_{\Delta\theta} \left(\frac{3}{2} \frac{\theta_5}{\theta_2} - 1 \right) \sigma_{ik} \frac{\partial \theta}{\partial x_k} + \varepsilon_{\Delta p} \frac{\theta_3}{2\theta_2} \theta \sigma_{ik} \frac{\partial \ln p}{\partial x_k} + \frac{\theta_4}{2\theta_2} \theta \frac{\partial \sigma_{ik}}{\partial x_k} \right.$$
$$\left. - \varepsilon_{\Delta\theta} \frac{\theta_4}{2\theta_2} \theta \omega \sigma_{ik} \frac{\partial \ln \theta}{\partial x_k} \right] + \varepsilon_{\Delta\theta} \frac{1}{\theta_2} \frac{25}{4} \frac{1}{\mathrm{Pr}^2} p \frac{\partial \theta}{\partial x_i} = -\frac{\varepsilon_q}{\mathrm{Kn}} \frac{1}{\theta_2} \frac{5}{2} \frac{1}{\mathrm{Pr}} \frac{p}{\mu} q_i. \quad (11)$$

The dimensional forms of these and all other equations presented in this paper can easily be obtained by setting $\varepsilon_\alpha = \mathrm{Kn} = 1$ where α stands for any of the subscripts that appear.

4. CHAPMAN–ENSKOG EXPANSION

Obviously, the multitude of coefficients complicates a proper analysis of the order of magnitude of the many terms arising. This is the reason why one has to assume that some of them are of comparable size. In the scaling employed for the Chapman–Enskog expansion, the CE scaling, all scaling coefficients but the

Knudsen numbers are assumed to be of order unity, so that

$$\varepsilon_t \simeq \varepsilon_v \simeq \varepsilon_\sigma \simeq \varepsilon_q \simeq \varepsilon_{\Delta\rho} \simeq \varepsilon_{\Delta\theta} \simeq \varepsilon_{\Delta p} = \mathcal{O}(1)\,.$$

In order to obtain scaling coefficents of order unity, one has to chose reference values ρ_0, v_0, θ_0 close to the maximum values that occur in the process.

The CE scaling is rather generous, as it neither restricts the values for v_i, σ_{ij}, q_i, nor those of the gradients of ρ, θ, p, and resolves time on a fine scale.

In order to simplify notation and expansion procedure we can set

$$\varepsilon_t = \varepsilon_v = \varepsilon_\sigma = \varepsilon_q = \varepsilon_{\Delta\rho} = \varepsilon_{\Delta\theta} = \varepsilon_{\Delta p} = 1. \tag{12}$$

This leads to the occurrence of the Knudsen number Kn as the sole dimensionless parameter (which appears in the denominator of the rhs only).

The Chapman–Enskog procedure[2,14] does not expand the conserved quantities ρ, θ, v_i, but only the non-conserved variables in a Knudsen number series. In our case, the only relevant variables are σ_{ij} and q_i for which one writes

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \mathrm{Kn}\sigma_{ij}^{(1)} + \mathrm{Kn}^2\sigma_{ij}^{(2)} + \cdots, \quad q_i = q_i^{(0)} + \mathrm{Kn}q_i^{(1)} + \mathrm{Kn}^2 q_i^{(2)} + \cdots \tag{13}$$

The expansion coefficients $\sigma_{ij}^{(\alpha)}$, $q_i^{(\alpha)}$ follow from inserting this ansatz into the set of 13 equations, and equating all terms with equal powers of the Knudsen number Kn. The details of this, including the careful treatment required for the time derivatives, are well explained in the cited literature, and thus are omitted here.

To zeroth order, the expansion yields

$$\sigma_{ij}^{(0)} = q_i^{(0)} = 0$$

which, when inserted into the conservation laws (1), results in the Euler equations.

The first order yields the well-known laws of Navier–Stokes and Fourier,

$$\sigma_{ij}^{(1)} = -2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}, \quad q_i^{(1)} = -\frac{5}{2}\frac{1}{\mathrm{Pr}}\mu\frac{\partial\theta}{\partial x_i}. \tag{14}$$

The second order Chapman–Enskog expansion gives the Burnett equations[2]

$$\sigma_{ij}^{(2)} = \frac{\mu^2}{p}\left[\varpi_1\frac{\partial v_k}{\partial x_k}S_{ij} - \varpi_2\left(\frac{\partial}{\partial x_{\langle i}}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_{j\rangle}}\right) + \frac{\partial v_k}{\partial x_{\langle i}}\frac{\partial v_{j\rangle}}{\partial x_k} + 2\frac{\partial v_k}{\partial x_{\langle i}}S_{j\rangle k}\right)\right.$$
$$\left. + \varpi_3\frac{\partial^2\theta}{\partial x_{\langle i}\partial x_{j\rangle}} + \varpi_4\frac{\partial\theta}{\partial x_{\langle i}}\frac{\partial\ln p}{\partial x_{j\rangle}} + \varpi_5\frac{1}{\theta}\frac{\partial\theta}{\partial x_{\langle i}}\frac{\partial\theta}{\partial x_{j\rangle}} + \varpi_6 S_{k\langle i}S_{j\rangle k}\right], \tag{15}$$

$$q_i^{(2)} = \frac{\mu^2}{\rho}\left[\theta_1 \frac{\partial v_k}{\partial x_k}\frac{\partial \ln\theta}{\partial x_i} - \theta_2\left(\frac{2}{3}\frac{\partial^2 v_k}{\partial x_k \partial x_i} + \frac{2}{3}\frac{\partial v_k}{\partial x_k}\frac{\partial \ln\theta}{\partial x_i} + 2\frac{\partial v_k}{\partial x_i}\frac{\partial \ln\theta}{\partial x_k}\right)\right.$$
$$\left. + \theta_3 S_{ik}\frac{\partial \ln p}{\partial x_k} + \theta_4 \frac{\partial S_{ik}}{\partial x_k} + 3\theta_5 S_{ik}\frac{\partial \ln\theta}{\partial x_k}\right]. \tag{16}$$

The third order expansion of the Boltzmann equation, or alternatively of its complete set of infinitely many moment equations, leads to the super-Burnett equations.[2] However, the system of 13 moment equations discussed presently does not contain enough information to yield these. Nevertheless, one could continue the expansion to higher orders, to obtain better and better approximations of the original Eqs. (2) and (3). When one considers an infinite number of expansion steps, the original equations might be recovered,[29,30] but the procedure becomes forbiddingly complicated.

The result of the CE expansion teaches us about the order of magnitude of stress and heat flux. For this, we note that the zeroth order contributions $\sigma_{ij}^{(0)}$ and $q_i^{(0)}$ vanish, and the leading order terms for both are of first order in Kn. Accordingly, in the Chapman–Enskog method, stress and heat flux turn out to be first order quantities in the Knudsen number. Thus, the original assumption of setting ε_σ and ε_q to unity allows values for stress and heat flux that are larger than the actual values. In other words, the CE expansion teaches us that instead of (12) we can set

$$\varepsilon_t = \varepsilon_v = \varepsilon_{\Delta\rho} = \varepsilon_{\Delta\theta} = \varepsilon_{\Delta p} = 1, \quad \varepsilon_\sigma = \varepsilon_q = \mathrm{Kn}. \tag{17}$$

It follows that ε_σ, ε_q cannot be chosen independently, but depend on the other chosen scales.

5. RESCALING AND ORDER OF MAGNITUDE APPROACH

The Chapman–Enskog expansion gave us information on the size of the scaling coefficients ε_σ, ε_q in the framework of the CE scaling. This will now be used in a different way to reduce the order of the original equations.

Before doing this, we introduce an alternative, and faster, approach to obtain the values for ε_σ and ε_q, which now follow from comparing the order of magnitudes of the various terms in Eqs. (10) and (11). Since all moments are dimensionless, their magnitude is expressed in the factors ε_α and Kn. We now employ the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms in the equation.* Accordingly, ε_σ and ε_q are bound by their possible maximum values, which follow from Eqs. (10) and (11) as will be outlined now.

In order to evaluate the principle, we rewrite Eq. (10) such that all terms containing ε_σ are on the left hand side (lhs) of the equation, and all other

terms on the right hand side (rhs). Similarly, Eq. (11) is rewritten such that all terms containing ε_q are on the lhs and all other terms on the rhs. Thus, in both equations the orders of magnitude of the terms on the left are unknown (since ε_σ and ε_q are unkown). From the above principle follows that the maximum term on the lhs must be equal to the maximum term on the rhs, so that

$$\max\left[\frac{\varepsilon_\sigma}{\mathrm{Kn}}, \varepsilon_\sigma\varepsilon_t, \varepsilon_\sigma\varepsilon_v\right] = \max[\varepsilon_v, \varepsilon_q, \varepsilon_q\varepsilon_{\Delta\theta}, \varepsilon_q\varepsilon_{\Delta p}],$$
$$\max\left[\frac{\varepsilon_q}{\mathrm{Kn}}, \varepsilon_q\varepsilon_t, \varepsilon_q\varepsilon_v\right] = \max[\varepsilon_{\Delta\theta}, \varepsilon_\sigma, \varepsilon_\sigma\varepsilon_{\Delta\theta}, \varepsilon_\sigma\varepsilon_{\Delta p}].$$

Since all orders of magnitude are positive, this yields the (coupled) conditions

$$\varepsilon_\sigma = \frac{\max[\varepsilon_v, \varepsilon_q, \varepsilon_q\varepsilon_{\Delta\theta}, \varepsilon_q\varepsilon_{\Delta p}]}{\max\left[\frac{1}{\mathrm{Kn}}, \varepsilon_t, \varepsilon_v\right]}, \quad \varepsilon_q = \frac{\max[\varepsilon_{\Delta\theta}, \varepsilon_\sigma 1, \varepsilon_\sigma\varepsilon_{\Delta\theta}, \varepsilon_\sigma\varepsilon_{\Delta p}]}{\max\left[\frac{1}{\mathrm{Kn}}, \varepsilon_t, \varepsilon_v\right]};$$

the further evaluation depends on the chosen scales. Choosing the basic scales as $\varepsilon_t = \varepsilon_v = \varepsilon_{\Delta\theta} = \varepsilon_{\Delta p} = 1$ we see

$$\varepsilon_\sigma = \frac{\max[1, \varepsilon_q]}{\max\left[\frac{1}{\mathrm{Kn}}, 1\right]}, \quad \varepsilon_q = \frac{\max[1, \varepsilon_\sigma]}{\max\left[\frac{1}{\mathrm{Kn}}, 1\right]}.$$

For Knudsen numbers below unity we thus obtain $\varepsilon_\sigma = \varepsilon_q = \mathrm{Kn}$, in agreement with the CE result (17). In other words, the CE method assigns the maximum possible values to the dependent scaling coefficients, and these values can be found by means of the above comparison without performing the CE expansion.

With the scaling (17) we still allow for fast flows and large gradients, and we resolve fine time scales. After introducing (17), the Eqs. (9)–(11) are rescaled as

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + v_k\frac{\partial\rho}{\partial x_k} + \rho\frac{\partial v_k}{\partial x_k} &= 0,\\
\rho\frac{\partial v_i}{\partial t} + \rho v_k\frac{\partial v_i}{x_k} + \frac{\partial p}{\partial x_i} + \mathrm{Kn}^1\frac{\partial\sigma_{ik}}{\partial x_k} &= 0,\\
\frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k} + p\frac{\partial v_k}{\partial x_k} + \mathrm{Kn}^1\left[\sigma_{ij}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \frac{\partial q_k}{\partial x_k}\right] &= 0,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\mathrm{Kn}^1\left[\frac{\partial\sigma_{ij}}{\partial t} + v_k\frac{\partial\sigma_{ij}}{\partial x_k} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + \left(\frac{\varpi_6}{\varpi_2} - 4\right)\sigma_{k\langle i}S_{j\rangle k}\right.\\
&\quad\left. + \frac{4}{5}\,\mathrm{Pr}\left(\frac{\varpi_3}{\varpi_2}\left(\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} - \omega q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}\right) + \frac{\varpi_4}{\varpi_2}q_{\langle i}\frac{\partial\ln p}{\partial x_{j\rangle}} + \frac{\varpi_5}{\varpi_2}q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}\right)\right]\\
&\quad + \mathrm{Kn}^0\frac{2}{\varpi_2}\frac{p}{\mu}\left[\sigma_{ij} + 2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\right] = 0,
\end{aligned} \tag{19}$$

$$\mathrm{Kn}^1\left[\frac{\partial q_i}{\partial t}+v_k\frac{\partial q_i}{\partial x_k}+q_k\frac{\partial v_i}{\partial x_k}+\frac{5}{3}q_i\frac{\partial v_k}{\partial x_k}+\frac{5}{2}\frac{1}{\mathrm{Pr}}\left(\frac{3}{2}\frac{\theta_5}{\theta_2}-1\right)\sigma_{ik}\frac{\partial\theta}{\partial x_k}\right.$$
$$\left.+\frac{5}{2}\frac{1}{\mathrm{Pr}}\left(\frac{\theta_3}{2\theta_2}\theta\sigma_{ik}\frac{\partial\ln p}{\partial x_k}+\frac{\theta_4}{2\theta_2}\theta\left(\frac{\partial\sigma_{ik}}{\partial x_k}-\omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k}\right)\right)\right]$$
$$+\mathrm{Kn}^0\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}\left[q_i+\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial x_i}\right]=0. \tag{20}$$

Now the equations are in a form where the proper order of magnitude—in the Chapman–Enskog sense—of *all* quantities is accounted for. The Knudsen number is the only dimensionless parameter in the system.

The assignment of the order of magnitude is the first step in the order of magnitude/order of accuracy method that was introduced in Refs. 2 and 24.

In the second step, the knowledge on the order of magnitude of all variables is used to construct a unique set of variables which has the minimum number of variables at a given order.[2,24,25] This step is not required for the present reduced set of equations.

The final step is to consider the terms in the equations according to their order of magnitude, and this will be outlined now. To be exact, we ask for the contributions to zeroth, first, second, and higher orders of stress and heat flux in the conservation laws.

At lowest—i.e. the zeroth—order, all terms with powers of the Knudsen numbers are ignored in the conservation laws (18). This yields the Euler equations,

$$\frac{\partial\rho}{\partial t}+v_k\frac{\partial\rho}{\partial x_k}+\rho\frac{\partial v_k}{\partial x_k}=0,$$
$$\rho\frac{\partial v_i}{\partial t}+\rho v_k\frac{\partial v_i}{x_k}+\frac{\partial p}{\partial x_i}=0, \tag{21}$$
$$\frac{3}{2}\rho\frac{\partial\theta}{\partial t}+\frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k}+p\frac{\partial v_k}{\partial x_k}=0.$$

Stress and heat flux do not appear, and the Eqs. (19) and (20) are not required.

At first order, all first order terms in the conservation laws are considered, and thus the complete conservation laws (18) must be considered together with the leading order expressions for stress and heat flux. The latter follow from the leading terms in (19) and (20) which are those of zeroth order. This gives, again, the laws of Navier–Stokes and Fourier,

$$\sigma_{ij}=-2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}},\quad q_i=-\frac{5}{2}\frac{1}{\mathrm{Pr}}\mu\frac{\partial\theta}{\partial x_i}.$$

Obviously, up to first order, the resulting equations agree with those from the Chapman–Enskog method. For higher orders, however, this is not the case. The

second order Chapman–Enskog expansion gives the Burnett equations (15) and (16), while in the order of magnitude method the second order requires to account for the zeroth and first order terms in Eqs. (19) and (20). Since there are no higher order terms, this step already recovers the complete original set of equations. No expansion is necessary, and the equations that result from the order of magnitude method are exact.

The same method was used on the infinite system of moment equations to derive the 13 moment equations for power potentials (1)–(3) that form the base of our discussion.[2,25] For Maxwell molecules, the method was applied to the infinite system of moment equations to third order; this yields the regularized 13 moment equations,[24] which were introduced in Refs. 27 and 28.

The order of magnitude method is a valuable alternative to the CE expansion. In particular all resulting sets of equations are stable, while the higher order Chapman–Enskog expansions lead to the unstable Burnett and super-Burnett equations.[9]

6. SLOW NON-ISOTHERMAL FLOW

6.1. SNIF Scaling

In order to illustrate the power of the order of magnitude method further, we now turn our attention to a different scaling of the equations. Following Ref. 31, we assume slow flows on a diffusion time scale, so that Knudsen number, Mach number and time scale coefficient are assumed to be small of the same order,

$$\mathrm{Kn} \simeq \varepsilon_M \simeq \varepsilon_t \simeq \mathcal{O}(\epsilon)$$

where now ϵ is the smallness parameter. To simplify notation, we write, similar to (12),

$$\mathrm{Kn} = \varepsilon_v = \varepsilon_t = \epsilon. \tag{22}$$

We shall be interested in two choices for the scale of the temperature gradient, $\varepsilon_{\Delta\theta} = 1$ and $\varepsilon_{\Delta\theta} = \epsilon$.

With this, the equations contain the two independent scaling parameters $\epsilon, \varepsilon_{\Delta\theta}$ which we are free to chose, and the additional dependent parameters $\varepsilon_{\Delta\rho}, \varepsilon_{\Delta p}, \varepsilon_\sigma, \varepsilon_q$ which will be related to $\epsilon, \varepsilon_{\Delta\theta}$ by arguments based on balancing orders.

To proceed, we require the equations with the scaling (22) which read

$$\begin{aligned} \epsilon\left[\frac{\partial\rho}{\partial t} + \varepsilon_{\Delta\rho} v_k \frac{\partial\rho}{\partial x_k} + \rho\frac{\partial v_k}{\partial x_k}\right] &= 0, \\ \epsilon^2\left[\rho\frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{x_k}\right] + \varepsilon_{\Delta p}\frac{\partial p}{\partial x_i} + \varepsilon_\sigma \frac{\partial\sigma_{ik}}{\partial x_k} &= 0, \end{aligned} \tag{23}$$

$$\epsilon\left[\frac{3}{2}\rho\frac{\partial\theta}{\partial t}+\varepsilon_{\Delta\theta}\frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k}+\rho\theta\frac{\partial v_k}{\partial x_k}\right]+\epsilon\varepsilon_\sigma\sigma_{ij}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}+\varepsilon_q\frac{\partial q_k}{\partial x_k}=0,$$

$$\begin{aligned}&\epsilon^2\varepsilon_\sigma\left[\frac{\partial\sigma_{ij}}{\partial t}+v_k\frac{\partial\sigma_{ij}}{\partial x_k}+\sigma_{ij}\frac{\partial v_k}{\partial x_k}+2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k}+\left(\frac{\varpi_6}{\varpi_2}-4\right)\sigma_{k\langle i}S_{j\rangle k}\right]\\&\quad+\epsilon\varepsilon_q\frac{4}{5}\mathrm{Pr}\left[\frac{\varpi_3}{\varpi_2}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}-\varepsilon_{\Delta\theta}\frac{\varpi_3}{\varpi_2}\omega q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}+\varepsilon_{\Delta p}\frac{\varpi_4}{\varpi_2}q_{\langle i}\frac{\partial\ln p}{\partial x_{j\rangle}}\right.\\&\quad\left.+\ \varepsilon_{\Delta\theta}\frac{\varpi_5}{\varpi_2}q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}\right]+\epsilon^2\frac{4}{\varpi_2}p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}=-\varepsilon_\sigma\frac{2}{\varpi_2}\frac{p}{\mu}\sigma_{ij},\end{aligned}\tag{24}$$

$$\begin{aligned}&\epsilon^2\varepsilon_q\left[\frac{\partial q_i}{\partial t}+v_k\frac{\partial q_i}{\partial x_k}+q_k\frac{\partial v_i}{\partial x_k}+\frac{5}{3}q_i\frac{\partial v_k}{\partial x_k}\right]\\&\quad+\epsilon\varepsilon_\sigma\frac{5}{2}\frac{1}{\mathrm{Pr}}\left[+\varepsilon_{\Delta\theta}\left(\frac{3}{2}\frac{\theta_5}{\theta_2}-1\right)\sigma_{ik}\frac{\partial\theta}{\partial x_k}+\varepsilon_{\Delta p}\frac{\theta_3}{2\theta_2}\theta\sigma_{ik}\frac{\partial\ln p}{\partial x_k}+\frac{\theta_4}{2\theta_2}\theta\frac{\partial\sigma_{ik}}{\partial x_k}\right.\\&\quad\left.-\varepsilon_{\Delta\theta}\frac{\theta_4}{2\theta_2}\theta\omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k}\right]+\epsilon\varepsilon_{\Delta\theta}\frac{1}{\theta_2}\frac{25}{4}\frac{1}{\mathrm{Pr}^2}p\frac{\partial\theta}{\partial x_i}=-\varepsilon_q\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}q_i.\end{aligned}\tag{25}$$

6.2. Finding the Values for $\varepsilon_\sigma, \varepsilon_q, \varepsilon_{\Delta p}$

As before, we need to find the order of magnitude for stress and heat flux which we determine by means of the approach of Sec. 5. We study Eqs. (24) and (25) where the order of all terms is explicit in the scaling parameters. The size (as measured by order of magnitude) of the dependent parameters cannot exceed that of the others, so that[2]

$$\varepsilon_\sigma=\frac{\max[\epsilon^2,\epsilon\varepsilon_q,\epsilon\varepsilon_q\varepsilon_{\Delta\theta}]}{\max[1,\epsilon^2,\epsilon\varepsilon_q]},\quad \varepsilon_q=\frac{\max[\epsilon\varepsilon_{\Delta\theta},\epsilon\varepsilon_\sigma\varepsilon_{\Delta\theta},\epsilon\varepsilon_\sigma^2,\epsilon\varepsilon_\sigma]}{\max[1,\epsilon^2]}.$$

For the evaluation we keep in mind that $\epsilon<1$ is the smallness parameter of the problem, and that $\epsilon\le\varepsilon_{\Delta\theta}\le 1$. Thus, the above criteria reduce to.

$$\varepsilon_\sigma=\max[\epsilon^2,\epsilon\varepsilon_q],\quad \varepsilon_q=\max\left[\epsilon\varepsilon_{\Delta\theta},\epsilon\varepsilon_\sigma\right].$$

Both conditions can only hold if

$$\varepsilon_\sigma=\epsilon^2,\quad \varepsilon_q=\epsilon\varepsilon_{\Delta\theta}.\tag{26}$$

[2] From the balance of momentum $(23)_2$ follows that $\varepsilon_{\Delta p}=\max[\varepsilon_\sigma,\epsilon^2]$, and we set $\varepsilon_{\Delta p}=\varepsilon_\sigma$ for the evaluation, since this is the more conservative choice.

The same result was obtained from a formal CE expansion, but this will not be presented here.

Thus, the balance of momentum $(23)_2$ assumes the form

$$\epsilon^2\left[\rho\frac{\partial v_i}{\partial t}+\rho v_k\frac{\partial v_i}{x_k}+\frac{\partial \sigma_{ik}}{\partial x_k}\right]+\varepsilon_{\Delta p}\frac{\partial p}{\partial x_i}=0.$$

Again we use the argument that a single term cannot be larger in size by one or several orders of magnitude than all others, from which follows that $\varepsilon_{\Delta p}\leq\epsilon^2$. To proceed we allow the maximum possible value, so that

$$\varepsilon_{\Delta p}=\epsilon^2.$$

It follows from (8) that $\varepsilon_{\Delta\rho}=\varepsilon_{\Delta\theta}$.

After these arguments, only the two smallness parameters ϵ and $\varepsilon_{\Delta\theta}$ are left in the equations, which now can be written as

$$\begin{aligned}
\epsilon\left[\frac{\partial\rho}{\partial t}+\rho\frac{\partial v_k}{\partial x_k}+\varepsilon_{\Delta\theta}v_k\frac{\partial\rho}{\partial x_k}\right]&=0,\\
\epsilon^2\left[\rho\frac{\partial v_i}{\partial t}+\rho v_k\frac{\partial v_i}{x_k}+\frac{\partial p}{\partial x_i}+\frac{\partial\sigma_{ik}}{\partial x_k}\right]&=0,\qquad (27)\\
\epsilon\left[\frac{3}{2}\rho\frac{\partial\theta}{\partial t}+\rho\theta\frac{\partial v_k}{\partial x_k}\right]+\epsilon\varepsilon_{\Delta\theta}\left[\frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k}+\frac{\partial q_k}{\partial x_k}\right]+\epsilon^3\sigma_{ij}\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}&=0,
\end{aligned}$$

$$\begin{aligned}
&\epsilon^4\left[\frac{\partial\sigma_{ij}}{\partial t}+v_k\frac{\partial\sigma_{ij}}{\partial x_k}+\sigma_{ij}\frac{\partial v_k}{\partial x_k}+2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k}+\left(\frac{\varpi_6}{\varpi_2}-4\right)\sigma_{k\langle i}S_{j\rangle k}\right]\\
&\quad+\epsilon^4\varepsilon_{\Delta\theta}\frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_4}{\varpi_2}q_{\langle i}\frac{\partial\ln p}{\partial x_{j\rangle}}+\epsilon^2\varepsilon_{\Delta\theta}^2\frac{4}{5}\,\mathrm{Pr}\left[\frac{\varpi_5}{\varpi_2}q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}-\frac{\varpi_3}{\varpi_2}\omega q_{\langle i}\frac{\partial\ln\theta}{\partial x_{j\rangle}}\right]\\
&\quad+\epsilon^2\varepsilon_{\Delta\theta}\frac{4}{5}\,\mathrm{Pr}\,\frac{\varpi_3}{\varpi_2}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}}+\epsilon^2\frac{2}{\varpi_2}\frac{p}{\mu}\left[\sigma_{ij}+2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}\right]=0,\qquad (28)
\end{aligned}$$

$$\begin{aligned}
&\epsilon^3\varepsilon_{\Delta\theta}\left[\frac{\partial q_i}{\partial t}+v_k\frac{\partial q_i}{\partial x_k}+q_k\frac{\partial v_i}{\partial x_k}+\frac{5}{3}q_i\frac{\partial v_k}{\partial x_k}\right.\\
&\quad\left.+\frac{5}{2}\frac{1}{\mathrm{Pr}}\left(\left(\frac{3}{2}\frac{\theta_5}{\theta_2}-1\right)\sigma_{ik}\frac{\partial\theta}{\partial x_k}-\frac{\theta_4}{2\theta_2}\theta\omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k}\right)\right]\\
&\quad+\epsilon^3\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{\theta_4}{2\theta_2}\theta\frac{\partial\sigma_{ik}}{\partial x_k}+\epsilon^5\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{\theta_3}{2\theta_2}\theta\sigma_{ik}\frac{\partial\ln p}{\partial x_k}\\
&\quad+\epsilon\varepsilon_{\Delta\theta}\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}\left[q_i+\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial x_i}\right]=0.\qquad (29)
\end{aligned}$$

6.3. Order of Magnitude Method ($\varepsilon_{\Delta\theta} = 1$)

To find the appropriate equations at the different orders, we use the order of magnitude method. We first consider the case of large temperature gradients, where $\varepsilon_{\Delta\theta} = 1$ and the conservation laws assume the form

$$\begin{aligned}
\epsilon\left[\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_k}{\partial x_k} + v_k\frac{\partial\rho}{\partial x_k}\right] &= 0,\\
\epsilon^2\left[\rho\frac{\partial v_i}{\partial t} + \rho v_k\frac{\partial v_i}{x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial\sigma_{ik}}{\partial x_k}\right] &= 0,\\
\epsilon\left[\frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \rho\theta\frac{\partial v_k}{\partial x_k}\right] + \epsilon\left[\frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k}\right] &= 0.
\end{aligned} \tag{30}$$

Since all terms in these equations carry at least the factor ϵ it follows that at order ϵ^0 all equations are identically fulfilled. In order to appreciate this result one has to recall that ϵ scales the derivatives of the variables rather than the variables themselves. At order ϵ^0 all derivatives can be ignored, so that this case refers to a constant state where $\rho = \rho_0$, $v_i = v_{i|0}$, $\theta = \theta_0$; ρ_0, $v_{i|0}$, θ_0 are constants.

At order ϵ^1 the momentum balance is still identically fulfilled, while the mass and momentum balance have contributions of first order. Thus, at the first glance we obtain the two equations

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_k}{\partial x_k} + v_k\frac{\partial\rho}{\partial x_k} &= 0,\\
\frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \rho\theta\frac{\partial v_k}{\partial x_k} + \frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k} &= 0.
\end{aligned}$$

These are not a closed system for the hydrodynamic variables, and an additional vector equation is required to determine velocity. From the scaled momentum balance $(30)_2$ we note in particular that the time derivative of velocity scales with ϵ^2. That is at order ϵ^1 velocity remains constant in time, which makes sense only if velocity is a constant, $v_i = v_{i|0}$; the constant $v_{i|0}$ vanishes in the restframe of the fluid. In this case, the balance equations reduce to those of an incompressible heat conducting fluid at rest, which must be furnished with the leading term of the equation for heat flux (29), viz.

$$\frac{\partial\rho}{\partial t} = 0, \quad \frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \frac{\partial q_k}{\partial x_k} = 0, \quad q_i = -\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial x_i}.$$

Bobylev in Ref. 31 performed a Chapman–Enskog expansion and at first order has a similar problem. He states: "To close this [first order] system [consisting essentially of mass and energy balance] it is necessary to add to it a single (vector) equation of the second order in ϵ that defines the evolution of the mean velocity." Contrary to Bobylev we prefer to use the zeroth order result for velocity–fluid at

rest–instead of the second order equation, which should only appear when second order is considered.

The balance Eqs. (27)–(29) reduce to a set of five equations for the five variables ρ, v_i, θ only if we include terms of first *and* second order in ϵ. The corresponding conservation laws are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_k}{\partial x_k} + v_k \frac{\partial \rho}{\partial x_k} &= 0, \\ \rho \frac{\partial v_i}{\partial t} + \rho v_k \frac{\partial v_i}{x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k} &= 0, \\ \frac{3}{2}\rho \frac{\partial \theta}{\partial t} + \rho\theta \frac{\partial v_k}{\partial x_k} + \frac{3}{2}\rho v_k \frac{\partial \theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k} &= 0, \end{aligned} \tag{31}$$

For consistency, this set must be closed by including expressions for σ_{ij} and q_i within *second* order, i.e. the leading order terms in the Eq. 28 for σ_{ij} (which are of order ϵ^2), and the first and second order terms in Eq. (29) for q_i. Since there are no second order terms in (29), it simply gives Fourier's law, so that stress and heat flux in the second order system are given by

$$\sigma_{ij} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + \frac{\mu^2}{p}\left[\varpi_3 \frac{\partial^2 \theta}{\partial x_{\langle i} \partial x_{j\rangle}} + \varpi_5 \frac{\partial \theta}{\partial x_{\langle i}} \frac{\partial \ln \theta}{\partial x_{j\rangle}}\right] \tag{32}$$

$$q_i = -\frac{5}{2}\frac{\mu}{\mathrm{Pr}} \frac{\partial \theta}{\partial x_i}. \tag{33}$$

Here, Fourier's law (33) was already used with (4) to eliminate the heat flux from the equation for σ_{ij}. The set of Eqs. (31)–(33) agrees with the equations given by Bobylev in Ref. 31.

At this point it is worthwhile to emphasize an interesting difference between our approach, and the arguments in Ref. 31 which employ the CE expansion. While the standard CE method expands only non-equilibrium quantities, it was necessary in Ref. 31 to expand also the equilibrium quantities ρ, v_i, θ, p. Our argument did not require this expansion of equilibrium quantities, but instead required the rescaling of the gradients in Sec. 3.2.

Indeed, Ref. 31 expands pressure in a CE series and distinguishes between a pressure $p_0(t)$ and the gradient of pressure, $\frac{\partial p}{\partial x_i}$. This is implicit in our results as well as will be discussed now. Our scaling showed that the gradient of pressure is small of second order, but nothing is said about the absolute value of pressure, which is non-zero. To obtain the Bobylev's result from (31)–(33), one can expand pressure into a Taylor series in space, where $p(x_i, t) = p_0(t) + \frac{\partial p}{\partial x_i}\delta x_i$, or—in dimensionless form—$\hat{p}(x_i, t) = \hat{p}_0(t) + \epsilon^2 \frac{\partial \hat{p}}{\partial \hat{x}_i}\delta \hat{x}_i$. This can be used to replace $p \to p_0$ when p appears as a factor, and one is only interested in lower order equations.

For the third order system, all terms up to third order must be considered, and this yields the full conservation laws (27) plus

$$\sigma_{ij} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \frac{2}{5}\,\mathrm{Pr}\,\frac{\mu}{p}\left[\varpi_3 \frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + \varpi_5 q_{\langle i} \frac{\partial \ln\theta}{\partial x_{j\rangle}} - \varpi_3 \omega q_{\langle i} \frac{\partial \ln\theta}{\partial x_{j\rangle}}\right], \quad (34)$$

$$\begin{aligned}
&\epsilon^2\left[\frac{\partial q_i}{\partial t} + v_k\frac{\partial q_i}{\partial x_k} + q_k\frac{\partial v_i}{\partial x_k} + \frac{5}{3}q_i\frac{\partial v_k}{\partial x_k}\right.\\
&\quad\left. + \frac{5}{2}\frac{1}{\mathrm{Pr}}\left(\left(\frac{3}{2}\frac{\theta_5}{\theta_2} - 1\right)\sigma_{ik}\frac{\partial\theta}{\partial x_k} - \frac{\theta_4}{2\theta_2}\theta\omega\sigma_{ik}\frac{\partial\ln\theta}{\partial x_k} + \frac{\theta_4}{2\theta_2}\theta\frac{\partial\sigma_{ik}}{\partial x_k}\right)\right]\\
&\quad + \frac{1}{\theta_2}\frac{25}{4}\frac{1}{\mathrm{Pr}^2}p\frac{\partial\theta}{\partial x_i} = -\frac{1}{\theta_2}\frac{5}{2}\frac{1}{\mathrm{Pr}}\frac{p}{\mu}q_i. \qquad (35)
\end{aligned}$$

Equation (34) for the stress is the same as (32), only that now the heat flux is not given by Fourier's law (33), but by the more complicated expression (35).

At fourth order, the full Eq. (28) must be used, while the heat flux is given by (29) without the term with the pressure gradient, which is of 5th order.

Obviously, the fifth order recovers the original Eqs. (27)–(29).

As in the CE scaling, no iterative process occurs in the method, and it recovers the full original set of equations after only few steps. The Chapman–Enskog method, on the other hand, would require an infinite number of steps in order to recover the original equations.

6.4. Order of Accuracy Method ($\varepsilon_{\Delta\theta} = \epsilon$)

Next, we turn the attention to an even more restrictive scaling that allows only moderate temperature gradients, so that $\varepsilon_{\Delta\theta} = \epsilon$. In order to have a complete set of equations for ρ, v_i, θ, we need to consider at least the second order terms in (27)–(29). By means of the same arguments as before, we obtain the set of second order as

$$\begin{aligned}
&\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_k}{\partial x_k} + \epsilon v_k\frac{\partial\rho}{\partial x_k} = 0,\\
&\rho\frac{\partial v_i}{\partial t} + \rho v_k\frac{\partial v_i}{x_k} + \frac{\partial\sigma_{ik}}{\partial x_k} + \frac{\partial p}{\partial x_i} = 0,\\
&\frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \rho\theta\frac{\partial v_k}{\partial x_k} + \epsilon\left[\frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k}\right] = 0,\\
&\sigma_{ij} = -2\mu\frac{\partial v_{\langle i}}{\partial x_{j\rangle}}, \quad q_i = -\frac{5}{2}\frac{\mu}{\mathrm{Pr}}\frac{\partial\theta}{\partial x_i},
\end{aligned}$$

These are just the Navier–Stokes–Fourier equations without the frictional heating term in the energy balance, which is ignored as a result of the chosen scaling.

At third order we find the full system of conservation laws (27) and some additional terms in the laws of Navier–Stokes and Fourier, viz.

$$\sigma_{ij} = -2\mu \frac{\partial v_{\langle i}}{\partial x_{j\rangle}} - \epsilon \frac{2}{5} \frac{\mu}{p} \Pr \varpi_3 \frac{\partial q_{\langle i}}{\partial x_{j\rangle}}, \quad q_i = -\frac{5}{2} \frac{\mu}{\Pr} \frac{\partial \theta}{\partial x_i} - \epsilon \frac{\mu\theta}{p} \frac{\theta_4}{2} \frac{\partial \sigma_{ik}}{\partial x_k}.$$

Higher order equations for this scaling can be constructed as before, but we shall not present any details. The equations at fifth order agree with the original set.

7. CONCLUSIONS

In this paper we have used the new order of magnitude expansion procedure to find approximations to a set of moment equations at different orders of a suitable smallness parameter. The new method recovers the original equations after a small number of steps. This stands in contrast to the classical Chapman–Enskog method, which leads to iterative approximations and requires an infinite number of steps to recover the original equations.

Our arguments show that use of scaling arguments in moment equations allows introduction of a large number of scaling parameters. All variables and their gradients are linked through the moment equations, and therefore the scaling parameters are linked as well. Accordingly, one can chose only some of the scaling parameters, e.g. the Knudsen number, while the values of others result from that choice by means of the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms in the equation.* Alternatively, the Chapman–Enskog expansion can be used to establish the order of magnitude of higher moments.

Several different ways of scaling were considered: (a) the original Chapman–Enskog scaling which employs only the Knudsen number as smallness parameter, (b) the scaling for slow non-isothermal flow, which adds smallness parameters for time scale and Mach number, and (c) a stronger version of (d) that also restricts the magnitude of temperature gradients.

The discussion of the scaling procedure shows that the Chapman–Enskog scaling is the least restrictive scaling choice, so that the results obtained from that scaling have the widest applicability. When this scaling is applied with the order of magnitude method to the 13 moment equations, one finds the Euler equations at zeroth order, the compressible Navier–Stokes equations at first order, and recovers the complete 13 moment equations at second order. The original CE expansion yields the Euler and Navier–Stokes equations as well, but at second order produces

the unstable Burnett equations, and cannot reproduce the original equations in a finite number of steps.

The more restrictive scaling choices give simpler variants of the above mentioned sets of equations, in particular the equations for slow non-isothermal flow. The original equations are recovered after five steps.

This paper presented the powerful order of magnitude method in an accessible setting, where its superiority over the Chapman–Enskog expansion can be evaluated easily. Both methods give the same results at lower orders, but only the order of magnitude method can recover the original equations after a finite number of steps, and gives stable equations at higher orders. The application of the method to the infinite system of moment equations for the Boltzmann equation was presented in Refs. 2,24 and 25.

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